Multiple soliton and bisoliton bound state solutions of the sine-Gordon equation and related equations in nonlinear optics

1973 J. Phys. A: Math. Nucl. Gen. 6 L112

(http://iopscience.iop.org/0301-0015/6/8/002)

J. Phys. A: Math., Nucl. Gen., Vol. 6, August 1973. Printed in Great Britain. © 1973

LETTER TO THE EDITOR

Multiple soliton and bisoliton bound state solutions of the sine-Gordon equation and related equations in nonlinear optics

P J Caudrey, J C Eilbeck†, J D Gibbon‡ and R K Bullough
Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, UK

Received 29 June 1973

Abstract. A prescription is given for obtaining mixed multiple soliton and bisoliton bound state solutions of the sine–Gordon equation and of related equations in nonlinear optics. Some applications in nonlinear optics and other branches of physics are sketched.

Recently multisoliton§ solutions of the sine–Gordon (SG) equation (Hirota 1972, Caudrey *et al* 1973a) and related nonlinear optics equations (Caudrey *et al* 1973b,c, Eilbeck *et al* 1973, Lamb 1973) have been discovered. The nature of these solutions when all the solitons have *real* amplitudes is now well understood. It was known that the two-soliton solution with a suitable choice of *complex* amplitudes gave a real composite pulse or bisoliton bound state. These pulse solutions were called 'mesons' for the SG equations (Perring and Skyrme 1962) and '0π' pulses for the self-induced transparency (SIT) equations (Lamb 1971). For conciseness we will use the generic name 'bion' for this sort of solution. The problem of choosing N_1+2N_2 complex coefficients in the general case to give N_1 solitons and N_2 bions was not solved: in this letter we report a solution of this problem and sketch some applications.

The original form of the N soliton solution involved a determinant with elements of the form

$$M_{ij}(x,t) = (a_i a_j)^{1/2}/(a_i+a_j)\{\exp(\theta_i) + (-1)^{i+j}\exp(-\theta_j)\} \tag{1}$$

where for the SG equation for example

$$\theta_i = \tfrac{1}{2}\{(a_i - a_i^{-1})t + (a_i + a_i^{-1})x\} + \delta_i. \tag{2}$$

A single bion is obtained by taking the two-soliton solution with the a_i and δ_i to be *antihermitian* pairs of complex constants. Although this choice gives a real solution the obvious generalization to multiple bion solutions gives a complex solution. To avoid this we consider the alternative but equivalent version of (1) (Caudrey *et al* 1973c)

$$M_{ij} = 2/(a_i+a_j)\cosh\{\tfrac{1}{2}(\theta_i+\theta_j)\}. \tag{3}$$

The transformation linking (1) and (3) involves a complex change in the δ_i. Equation (3), with the choice of a *hermitian* pair of complex constants in the two-soliton solution, gives the same bion solution. However the generalization to multiple bion solutions is

† Address after 1st October 1973: Department of Mathematics, Heriot-Watt University, Edinburgh, UK.
‡ Address after 1st October 1973: Istituto di Fisica dell' Università, Pavia, Italy.
§ A review of the subject of solitons has been given by Scott *et al* (1973).

now real since M_{ij} is symmetric. The solution to the multiple soliton/bion problem can now be stated: take the N_1+2N_2 soliton solution with the alternative determinant element (3); choose N_1 real constants $a_{n,1}$ and N_2 pairs of complex constants $a_{n,2}$ and $a_{n,2}{}^*$. The same prescription applies to both the SG solutions and the nonlinear optics equations solutions.

The bion pulse is a single localized disturbance with internal oscillations. It has the same 'collision stability' as ordinary solitons, in both bion–bion and bion–soliton collisions. Both the 'envelope' and the 'oscillations' of the bion suffer a phase change in the collision, and a multiple collision is equivalent to a sequence of two bion/soliton collisions. Some applications of these remarkable solutions are sketched below, and will be reported in detail elsewhere.

The SG equation has been used as a simple classical field theory model for elementary particles (Perring and Skyrme 1962, Rubinstein 1970). In this model a soliton is a 'fundamental particle' or 'kink' and a bion is a 'meson'. Our results show that in this model a meson collides elastically with both other mesons and with kinks. Although the lagrangian is nonlinear, the phase shifts resulting from a multiple collision are linear sums of two-body phase shifts!

The SIT equations describe the evolution of the *envelope* of a resonant carrier wave interacting with a medium of two-level atoms (Caudrey *et al* 1973b, Eilbeck *et al* 1973, Lamb 1971, 1973). The soliton pulse is known as a '2π' pulse and the bion as a '0π' pulse (the numbers refer to the time area of the pulse). Our results enable us for the first time to generate pulses which split up into N_1 '2π' pulses and N_2 '0π' pulses. As an example we plot in figure 1 a pulse which breaks up into a '2π' pulse and a '0π' pulse. As $t \to \pm\infty$ this pulse is described by the equation

$$E(x,t) \underset{t\to\pm\infty}{\to} E_3 \operatorname{sech}\theta_3{}^\pm + \frac{2E_R \operatorname{sech}\theta_R{}^\pm(\cos\theta_I{}^\pm - \eta\sin\theta_I{}^\pm \tanh\theta_R{}^\pm)}{1+\eta^2\sin^2\theta_I{}^\pm \operatorname{sech}^2\theta_R{}^\pm} \tag{4}$$

where

$$E_1 = E_2{}^* = E_R + \mathrm{i}E_I, \qquad \eta = E_R/E_I \tag{5a}$$

$$\theta^\pm = \theta + \gamma^\pm$$

$$\gamma_R{}^+ - \gamma_R{}^- = \ln[\{(E_R - E_3)^2 + E_I{}^2\}/\{(E_R+E_3)^2 + E_I{}^2\}] \tag{5b}$$

$$\gamma_I{}^+ - \gamma_I{}^- = 2\tan^{-1}\{E_I/(E_R - E_3)\} - 2\tan^{-1}\{E_I/(E_R+E_3)\} \tag{5c}$$

$$\gamma_3{}^+ - \gamma_3{}^- = 2(\gamma_R{}^+ - \gamma_R{}^-) \tag{5d}$$

The first term on the right-hand side of (4) is a soliton and the second term is a bion. Note that 'pulse break-up' is simply a 'pulse collision' at $t = 0$ viewed at times $0 \leqslant t < \infty$; the initial pulse can be constructed by the collision of two or more pulses previously injected into the medium, or by injecting the initial pulse shape itself directly into the system at $t = 0$.

Another set of nonlinear optics equations with similar multisoliton solution are the so called reduced Maxwell–Bloch (RMB) equations (Caudrey *et al* 1973b, c, Eilbeck *et al* 1973), an approximate version of the full MB equations, appropriate at low densities. The RMB bion solution is (Eilbeck *et al* 1973) the generalized version of the '2π' solution of the SIT equations. The carrier wave (the oscillitory part of the bion) need not be resonant in the RMB equations, and each bion can have a *different* carrier frequency. In the language of the SIT equations we can start with a '$2N\pi$' pulse with a carrier wave modulation which breaks up into N '2π' pulses, each with a different carrier frequency.

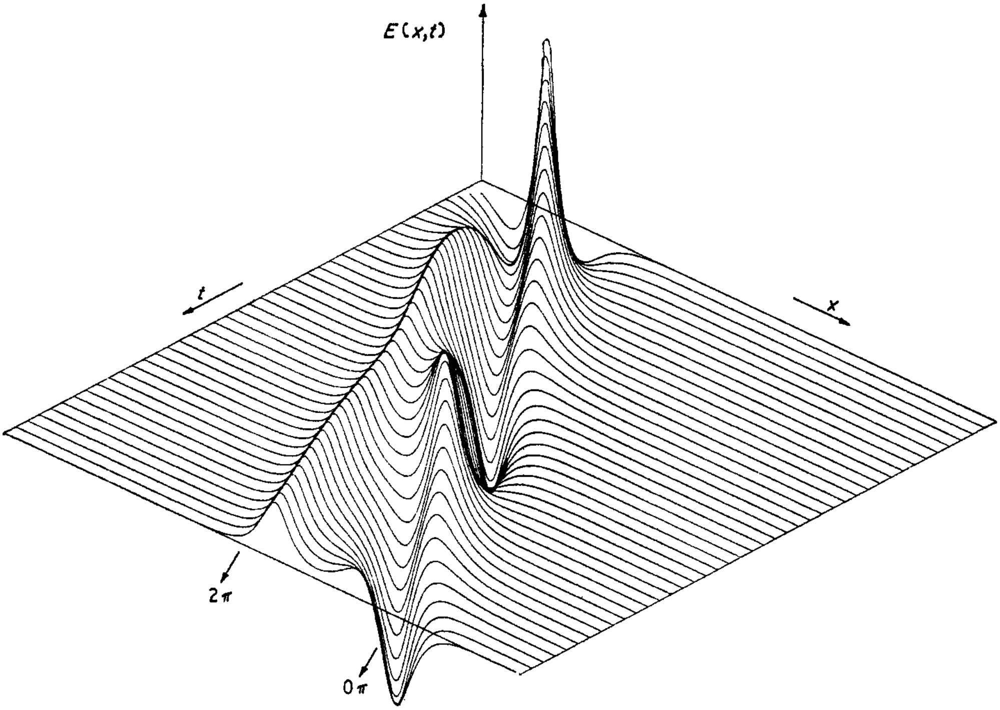

Figure 1. Three-soliton pulse solution of the SIT equations splitting up into a '2π' pulse and a '0π' pulse.

Bion solutions are important from a mathematical viewpoint as they shed light on the complicated theories (Whitham 1970, and references therein) of nonlinear wave propagation. For small amplitudes the bion is a stable (Eilbeck *et al* 1973) slowly varying pulse envelope modulating a carrier wave. Our results suggest that for the general initial value problem the envelope forms a series of stable pulses, rather than a shock (for the equations considered here, at least) and that a dispersive term is needed in the general nonlinear dispersion relation for the carrier wave. A similar conclusion has been reached by Chu and Mei (1970, 1971) in the case of Stokes waves using a WKB perturbation technique.

One of us (JCE) would like to thank the SRC for the financial support of a research assistantship.

References

Caudrey P J, Eilbeck J C and Gibbon J D 1973c *UMIST Maths. Dept. Preprint* TP/73/5
Caudrey P J, Gibbon J D, Eilbeck J C and Bullough R K 1973a *Phys. Rev. Lett.* **30** 237–8
—— 1973b *J. Phys.* A: *Math., Nucl. Gen.* **6** L53–6
Chu V C and Mei C C 1970 *J. Fluid Mech.* **41** 873–87
—— 1971 *J. Fluid Mech.* **47** 337–51

Eilbeck J C, Gibbon J D, Caudrey P J and Bullough R K 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** in the press
Hirota R 1972 *J. Phys. Soc. Japan* **33** 1459–63
Lamb G L Jr 1971 *Rev. mod. Phys.* **43** 99–124
—— 1973 *Coherent Optical Pulse Propagation as an Inverse Problem* to be published
Perring J K and Skyrme T M R 1962 *Nucl. Phys.* **31** 550–5
Rubinstein J 1970 *J. math. Phys.* **11** 258–66
Scott A C, Chu F Y F and McLaughlin D W 1973 *Proc. IEEE* to be published
Whitham G B 1970 *J. Fluid Mech.* **44** 373–95